# FRACTIONAL CALCULUS APPLICATIONS IN CONTROL SYSTEMS

Mark Axtell
Veda Incorporated
5200 Springfield Pike, Suite 200
Dayton, OH, 45431

Michael E. Bise
Veda Incorporated
5200 Springfield Pike, Suite 200
Dayton, OH, 45431

## Abstract

Standard control systems can be characterized by type in the *s*-domain. Typically these types are of integer order. This paper explores some of the implications of non-integer order systems in the s-domian. In order to accomplish this results from the area of fractional calculus, which defines mathematics of non-integer order derivative and integration are utilized.

## Introduction

In general, fractional calculus analysis techniques have not been widely incorporated into the engineering sciences. This is partly because of the conceptually difficult idea of taking the 1/2 derivative and its notorious lack of geometrical interpretation, and partly because there are so few physical applications.

Control systems engineers have made extensive use of operational calculus techniques in the design and analysis of complex networks. Computational simplifications have been made through the use of the Laplace Transform, $L\{f(t)\}$, operator which moves the system analysis from the time domain to the s-domain.

$$L\{f(t)\} \Rightarrow F(s)$$

One of the great advantages of the Laplace transform is that fundamental mathematical operations like convolution, differentiation and integration, which frequently introduce computational difficulties in the time domain, are found to be simple algebraic operations in the *s*-domain. For this reason, much of control systems design and analysis is done exclusively in the *s*-domain, returning to the time domain only to verify those system design parameters that are specified in time( e.g. rise time, peak overshoot, settling time, etc.). While the *s*-domain is certainly more analytically robust, some time domain operations and techniques for which no Laplace transform exists or is thought to exist, have remained absent from the control systems engineering literature. Fractional calculus is one example.

The phrase "fractional calculus" is immediately disconcerting and seems contrary to the existing theory of calculus. But conceptually it is quite simple. Integration and differentiation to non-integer order. Consider the following :

let f(x) be some function : $x \Rightarrow R^1$

let $$g(x) = \frac{d(f(x))}{dx} = f^1(x) \; ; \qquad (1)$$

then $$g_m(x) = \frac{d^m f(x)}{dx^m} = f^m(x) \qquad (2)$$

where m is some integer (1,2,3,4,5...). Typically $f^m(x)$ is referred to as the $m^{th}$ derivative of f(x). The notion of fractional calculus is to remove the integer restriction on m. That is, let m be a rational number (fraction) q. Rewriting (2) in this way yields:

$$g_r(x) = \frac{d^q f(x)}{dx^q} = f^q(x) \qquad (3)$$

where r is a real number. So a fractional derivative of order 1/2 would be:

$$g_{1/2}(x) = \frac{d^{1/2} f(x)}{dx^{1/2}} = f^{1/2}(x) \qquad (4)$$

This is known as a *semi-derivative.*

The idea of fractional calculus is not new. In 1695 Leibniz discussed it in a letter to L'Hospital. But it was contributions by Liouville (1832), Abel(1823), Heaviside(1892) and Riemann(1953), that formalized the theory of the non-integer order, $q^{th}$, derivative.

The area of fractional calculus has primarily been the domain of mathematicians and has a firm and long standing theoretical foundation. The purpose here is not to reformulate or prove the existence of non-integer order derivatives(or integrals) but to accept their existence and detail some of the interesting results which have applications in control theory. Consequently the present treatment of the fractional calculus (in the time -domain) will be decidedly non-rigorous and is provided for introductory and notational purposes only. (For further detail see references). The objective here is to confine the analysis to the *s*-domain

Several formulations and notational methods have been developed for fractional calculus. One of the most useful definitions of the $q^{th}$ order derivative is

$$\frac{d^q f}{dx^q} = \frac{1}{\Gamma(-q)} \int_0^x \frac{f(y)\,dy}{[x-y]^{q+1}}, \quad q < 0, \qquad \textbf{(5)}$$

Note that by defining the derivative operator in this way, the fractional integral operation turns out to be:

$$\frac{d^q}{dx^q}, \quad \text{for } q < 0 \qquad (6)$$

So that $$\int f(x)dx = \frac{d^{(-1)}}{dx^{(-1)}}, \qquad (7)$$

All of this may seem quite confusing at first but it is actually quite an intuitively succinct scheme which demonstrates the inverse relationship between integration and



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differentiation. The expression in (6) is a de facto generalized representation for the standard calculus operators and is valid for both positive(differentiation) and negative(integration) values of q. The expression in (6) is known in the literature as a differintegral operator, and can often be examined as a stand alone operator, without regard to the sign of q.

Differintegrals, in general, do not behave like their integer ordered counterparts. While they do have some properties, such as linearity, in common with the standard calculus operators, many properties and operations do not translate so simply. The chain rule and composition get terribly complicated in this paradigm. Fortunately one of the features most important to control systems engineers, the Laplace transform is very straightforward.

Recall the Laplace transform of the derivative or integrator operator :

$$L\left\{\frac{d^q f}{dx^q}\right\} = s^q L\ \{f\} - \sum_{k=0}^{q-1} s^k \frac{d^{q-1-k} f}{dx^{q-1-k}}(0), \qquad \textbf{(8)}$$

$$q = 0, \pm 1, \pm 2, \ldots .$$

$$L\left\{\frac{d^q f}{dx^q}\right\} = s^q L\ \{f\} - \sum_{k=0}^{n-1} s^k \frac{d^{q-1-k} f}{dx^{q-1-k}}(0), \qquad \text{all } q, \qquad (9)$$

This is an extremely useful result and has interesting mathematical implications for control systems engineers. In this paper we will examine some simple applications of the differintegral to the conventional control system problems.

## Fractional Type Control Systems

In order to investigate the utility of the differintegral in systems theory, it is first necessary to develop a system paradigm that can accommodate the mathematical peculiarities that will be induced by the differintegral operator. Fortunately this can be accomplished by considering a standard linear systems model and relaxing certain restrictions with regard to integer order. The end product of this will be a general system model that can be used to examine the effects of differintegrals in the context of conventional control theory.

Equation (9) shows the relationship between time-domain differintegrals and their *s*-domain counterparts. This result is intuitively reassuring and simplifies the analysis effort greatly. By (Laplace) transforming the differintegral operator into the s-domain, complicated manipulations of the gamma function, $\Gamma(x)$, can be reduced to simple algebraic manipulations of the $s^q$ operator. Additionally, since many familiar control systems concepts and methods are defined exclusively in the s-domain, the following analysis is concerned primarily with issues in this domain.

The first step is to define a suitable s-domain model. Consider the simple unity gain feedback control system which is shown below :

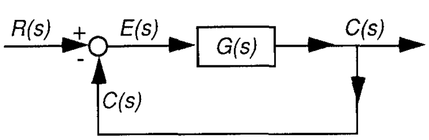



**FIGURE 1**

For this system the open loop transfer function is defined as

$$G(s)H(s) = \frac{K \prod_{h=1}^{w}(s - z_h)}{s^m \prod_{c=1}^{u}(s - p_c)} \qquad \textbf{(10)}$$

The *type* of the system, which plays a distinct role in determining system dynamics, is defined by the value of m in (10). Functionally, $s^m$ *is an s-domain representation of* m cascaded time-domain integrators. Letting m assume a non-integer value, q, allows (10) to be rewritten as

$$G(s)H(s) = \frac{K \prod_{h=1}^{w}(s - z_h)}{s^q \prod_{c=1}^{u}(s - p_c)} \qquad \textbf{(11)}$$

And the resulting system, (11) is said to be of *fractional type.* The type of a system is a standard classification used to segregate system transfer functions based on some particular performance measure (e.g. steady state error, etc.). By defining a *fractional type* it is possible, in some instances, to identify intermediate types which bridge what are considered to be discrete classes. This , of course stems from the ability to continuously vary q in (11), which *is a direct consequence of the fractional order of the* differintegral.

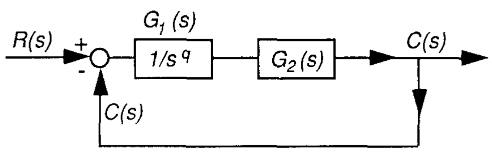
## The $s^q$ Operator.

The system described in (11) is characterized by the sq term in the denominator. Keep in mind that the sign of q can change (differentiation vs. integration) without introducing any structural difficulties into the model.

Mathematically $s^q$ is complex and can be expressed as

$$s^q = |s|^q e^{jqw} \qquad \textbf{(12)}$$

This is a standard result from complex analysis. Separating out the fractional part in fig.1 :




$G_1(s) = |G_1(s)| \angle \Theta(s)$. For q =1 , $G_1(s)$ is a simple (pure) integrator. As q decreases toward 0, the effects of the integration operation diminish accordingly. For q: 0<q<<1, $s^q$ can be considered a *weak* integrator. Ultimately at q=0 *all the effects of the integration operation have been* eliminated completely since $s^0 = 1$. At this point G1 could be removed since the the gain and phase parts of (12) have gone to 1 and 0 respectively (i.e. $1 \angle 0$).
What are the effects of integration referred to above? Typically addition of a pure integrator to a system is accompanied by a slowing down of system response, among other things. By adjusting q , some measure of control can be gained over *how much* the system response slows down.

Of course an analogous situation exists for q in the numerator except that in this case the time domain operation is derivation instead of integration . For fractional calculus operators defined as in (8), this just means changing the sign of q.

The ability to vary q in this way can significantly affect control system performance and is the central property of the fractional calculus that we seek to exploit. It is especially interesting to set q equal to some very small positive value and observe which properties of sq are retained and which are diminished. In the following sections some fundamental control theory results are reexamined within the context of this varying q.

## Stability and Control Impacts of $s^q$

It is well known that the locations of the poles and zeros of G(*s*) (unity gain feedback) determine the roots of the characteristic equation of a system and consequently influence its stability. By changing the type of the system (i.e. varying q), stability can be modified. Consider

$$G(s) = \frac{K(s - z_1)(s - z_2) \cdots (s - z_w)}{s^q(s - p_1)(s - p_2) \cdots (s - p_v)} \qquad \mathbf{(13)}$$

For q=1, an open loop pole exists at s=0. The existence of this pole is guaranteed for all values of q >0. Notice that even when $0 < q << 1$ there is a pole at zero since $s^q = 0$ has a solution at 0 for all q. To see what effect the fractional value of q has on system stability, consider the root locus plot of (13).

For a sample point, $s_1$, to be on the root locus of the system, $s_1$ must satisfy the following equation.

$$1 + G(s) = 0 \qquad (14)$$

or equivalently

$$|G(s)| = 1 , \qquad (15)$$

$$\angle G(s) = \beta = (1+2h)180, \ (h=0, \pm1,..)$$

Combining (13) and (15)

$$|K| = \frac{|s^q|\,|s - p_1|\,|s - p_2| \ldots |s - p_v|}{|s - z_1|\,|s - z_2| \ldots |s - z_w|}$$

and

$$\beta = q\angle s + \angle s\text{-}p_1 + \ldots \angle s\text{-}p_v - \angle s\text{-}z_1 - \ldots \angle s\text{-}z_w$$

Notice once again as q approaches zero the effect of sq on |K| and β decreases. Obviously by choosing q small enough, the fractional pole at 0 can be reduced to $1 \angle 0$, and is thereby made transparent to the rest of the system. So as q is reduced from 1 to zero, the fractional pole at the origin can be adjusted to exhibit different *degrees* of pole-like behavior. For example a *strong* pole (q=1) tends to shift the root locus to the right which may introduce instability. A *weak* pole on the other hand, (q=1/n,n large) behaves like a linear multiplier of value 1. This suggests that the tendency of a pole to shift the entire root locus plot can now be scaled to accommodate a particular application.

Once again an equivalent analysis can be performed for a system fractional zero by merely changing the sign of q. More extensive analysis which examines the combinatorial effects of fractional poles and zeros, and their relationship to system damping coefficients and other performance criteria are beyond the scope of this work.

## Frequency Response

The utility of the being able to vary q continuously is apparent when considering system frequency response. In this section a conventional Bode analysis will be performed on a simple system which incorporates a fractional element. Consider

$$G(jw) = \frac{K_m(\alpha+jw)}{(jw)^q(\beta+jw)} \qquad (16)$$

which can be expressed equivalently as,

$$|G(jw)| = |K_m| + |\alpha+jw| - q|jw| - |\beta+jw| \qquad (17)$$

$$\angle G(jw) = \angle K_m + \angle(\alpha+jw) - q\,90 - \angle(\beta+jw) \qquad (18)$$

Here the advantage of fractional q is more obvious. Recall that when q=1, (*jw*) is considered to have a constant phase characteristic of 90 degrees. By introducing a fractional q, equation (18) shows that an arbitrary phase shift can be obtained by adjusting the value of q. Moreover , this phase shift is independent of frequency, w. By choosing q between 0 and 4, the phase response of the system can be precisely offset (increased or decreased) by any amount from 0 to 360 degrees.

The following example demonstrates this behavior more clearly,

Define a unity gain feedback system, G(*s*) ,

$$G(s) = \frac{10\,(s+2)}{s^q\,(s+1)} \qquad (19)$$

then

$$G(jw) = \frac{10\,(jw+2)}{jw^q\,(jw+1)} \qquad (20)$$

Figures 2 & 3 below show the gain and phase response,respectively, as functions of frequency and q.

**GAIN RESPONSE FOR SELECTED q**

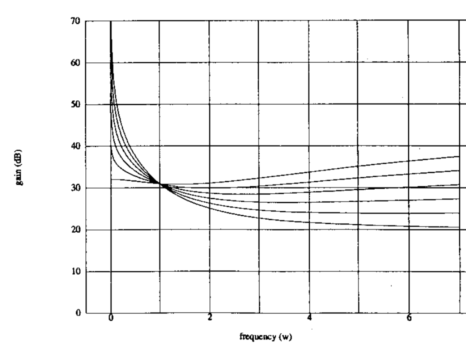



**FIGURE 2**

**PHASE RESPONSE FOR SELCETED q**

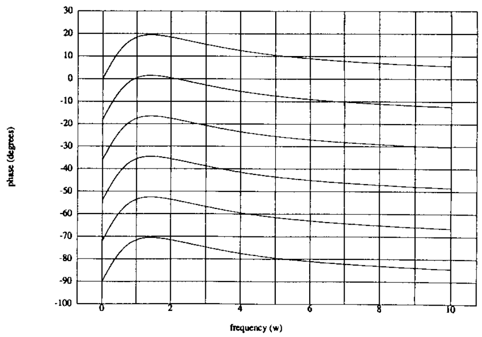



**FIGURE 3**

Figure 2 shows that as the type of the system ,q, is reduced from 1 to 0, the effect is to reduce the system gain at a constant rate, *-qw*. Figure 3 demonstrates that a fractional pole can be used to add phase in a linear fashion. In this figure the bottom trace, -90 degrees at $w$=0, is the standard response of a strong pole located at the origin. As q is reduced, phase is added, and the whole response is shifted upward. This is an interesting result when contrasted with conventional lead, lag networks or PID-type controllers which can have undesirable phase-gain trade offs at certain frequencies.

## Steady State Error and $s^q$

By now it is evident that the dynamics of fractional type operators are tied closely to system properties that exhibit distinct *qualitative* differences between conventional (i.e. integer) system types.(e.g . TYPE 0, TYPE 1, etc.). Another performance measure that fits this category is the steady state error.

For the simple unity gain feedback system considered above , the steady-state error, $e_{SS}$, is defined as,

$$e(t)_{ss} = \lim_{s \to 0} s \left[ \frac{s_m(1 + T_a s)(1 + T_b s) \cdots R(s)}{s_m(1 + T_a s)(1 + T_b s) \cdots + K_m(1 + T_a s)(1 + T_b s) \cdots} \right] \tag{21}$$

In the limit, $s^q$ goes to zero if q is fractional or integral. Of course this is dependent on the input R($s$) as well. The following table compares the effects of fractional and integral $s^q$ on steady state error.

| System Type, q | Input | $Error_q$ | $Error_q$ | |
|---|---|---|---|---|
| 0<q<1 | Unit step | 0 | Ro/1 + Ko | Type 0 |
| | Ramp | 00 | 00 | |
| | Parabolic | 00 | 00 | |
| 1<q<2 | Unit step | 0 | 0 | Type 1 |
| | Ramp | 0 | $R_1/K_1$ | |
| | Parabolic | 00 | 00 | |
| 2<q<3 | Unit step | 0 | 0 | Type 2 |
| | Ramp | 0 | 0 | |
| | Parabolic | 0 | $R_2/K_2$ | |

**TABLE 1**

Obviously since the fractional value of system q does not exactly cancel that the integer powers of $s$ for the various inputs , there will always be a fractional order s term in the numerator or the denominator which accounts for the fact that the error is always either 0 or infinite.

Earlier it was discussed that for a type q system, q: 0<q<<1, the root locus plot would resemble that of a TYPE 0 system, since the weak pole at $s^q$ has little effect. However from Table 1 it is apparent that with a unit step input, the steady state error of the system is 0, a characteristic usually found in a system of at least TYPE 1.Once again , a fractional type system seems to combine some of the characteristics of TYPE N and (N+1) systems.

## Conclusion/Recommendations

Fractional calculus operators, Laplace transformed differintegrals , exist. They have properties and behave differently from their standard integer-order counterparts. This brief survey has provided several examples of simple applications in the control sciences. All the analysis was performed in the $s$-domain in order to simplify the explanations without all the computational rigor that would be required in the time domain. Additionally, by staying in the $s$-domain attention can be focussed on examining widely known elementary control laws and their relationships and determining what effects fractional q has upon them.

Here the differintegral operator was introduced in a system - theoretic context by making use of a result involving the Laplace transform. Extensive research is still in required in order to identify all the impacts that fractional order systems can have in the controls field.

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